2D frequency-domain elastic full-waveform inversion using a $P_0$ finite volume forward problem

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SUMMARY

We present a 2-D frequency-domain elastic full-waveform inversion algorithm based on a forward problem solved with a frequency-domain parsimonious $P_0$ finite-volume method. A preconditioned conjugate gradient method allows the reconstruction of elastic parameters for various acquisition configurations. Solving many times the forward problem is required in the main steps of the inversion algorithm. In the frequency domain, the forward problem reduces to the resolution of a huge and sparse linear system which is efficiently performed with a massively parallel direct solver. The designed algorithm is validated with three simple synthetic examples for the reconstruction of both $P$ and $S$ wave velocities from vertical and horizontal particle velocities in regular equilateral triangular meshes.

INTRODUCTION

Quantitative seismic imaging of elastic parameters is one of the main challenge of geophysical exploration at different scales (subsurface, oil exploration, crustal and lithospheric investigations). Frequency-domain full-waveform inversion (FWI) (Pratt and Worthington, 1990; Pratt et al., 1996, 1998) allows to build accurate velocity models of complex structures from long offset acquisition geometries using only few discrete frequencies thanks to the wavenumber redundancy provided by multi-aperture geometries. Moreover, proceeding sequentially from the low frequencies to the high ones defines a multisolution imaging strategy which helps to fulfill the assumptions underlying local optimization approaches. Applications to real data using the acoustic approximation for 2D geometries have been performed with success for imaging complex structures (Ravaut et al., 2004; Operto et al., 2006), while the reconstruction of elastic parameters has been found to be a quite challenging problem (Gelis et al., 2007) mainly due to the high numerical cost of the forward problem. Recently, a 2D parsimonious $P_0$ Finite Volume (FV) method has been proposed by Brossier et al. (2007) for accurate and efficient elastic wave modeling on triangular meshes. Accurate modeling of wave propagation for complicated topographies is achieved for a discretization of 15 triangular cells per minimum wavelength which leads to significant memory and CPU-time savings compared to that required by $O(\Delta x^2)$ Finite Difference (FD) methods. In this study, we present a massively parallel frequency-domain FWI algorithm for imaging 2D elastic parameters based on the $P_0$ FV forward problem and solved using a massively parallel direct solver. The algorithm is validated with three simple synthetic examples.

THEORY

Forward problem

The 2D elastic P-SV wave equation in the frequency domain is described by the following first order velocity-stress system:

$$
-\iota \omega V_x = \frac{1}{\rho(x)} \left( \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xz}}{\partial z} \right) + F_x
$$

$$
-\iota \omega V_z = \frac{1}{\rho(x)} \left( \frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{zz}}{\partial z} \right) + F_z
$$

$$
-\iota \omega \sigma_{xx} = (\lambda(x) + 2\mu(x)) \frac{\partial V_x}{\partial x} + \lambda(x) \frac{\partial V_z}{\partial z} - \iota \omega \sigma_{x0}
$$

$$
-\iota \omega \sigma_{zz} = \lambda(x) \frac{\partial V_x}{\partial x} + (\lambda(x) + 2\mu(x)) \frac{\partial V_z}{\partial z} - \iota \omega \sigma_{z0}
$$

$$
-\iota \omega \sigma_{xz} = \mu(x) \left( \frac{\partial V_x}{\partial x} + \frac{\partial V_z}{\partial z} \right) - \iota \omega \sigma_{x0},
$$

where $(V_x, V_z)$ are the particles velocities, $(\sigma_{xx}, \sigma_{xz}, \sigma_{zz})$ the stresses, $\lambda, \mu$ are the Lamé coefficients, $\rho$ the density and $\iota \omega \sigma_0$ the angular frequency. External excitations are given by $(F_x, F_z)$ for forces and $(\sigma_{x0}, \sigma_{z0}, \sigma_{x0})$ for stresses. Perfectly Matched Layers (Berenger, 1994) are introduced in system 1 as absorbing conditions to avoid parasite reflections from grid boundaries.

Medium is discretized with triangular cells and a parsimonious $P_0$ FV method is applied to system 1 (Brossier et al., 2007). Triangular description of medium allows to model accurately complex topographies without the staircase description commonly used in classical FD methods. We finally end up with a linear system which can be recast in a matrix form, with two particles velocities unknowns per triangular cell:

$$
AV = S
$$

where the complex-value impedance matrix $A$ depends on the frequency and the medium properties.

The $P_0$ FV approach leads to compact spatial stencil similar to that of $O(\Delta x^2)$ FD methods which limits the numerical bandwidth of the matrix and hence its fill-in during LU factorization compared to higher order ones (Hustedt et al., 2004). The $P_0$ FV approach requires 10 cells per minimum shear wavelength to obtain acceptable dispersion properties when horizontal free surface is considered and 15 cells when complex topographies are considered for propagation distances lower than 100 wavelengths. Due to the low order of $P_0$ FV, only regular equilateral meshes provide solutions with an acceptable accuracy. Unstructured meshes do not provide enough accurate solutions for FWI applications.

The LU factorization, although memory demanding, allows to solve efficiently thousands of forward problems since the factorization is independent of the right-hand side source terms in the equation 2. In this study, we used the massively parallel direct solver MUMPS which has been developed for distributed-memory platform (Amestoy et al., 2006).

Frequency-domain full-waveform inversion

The inverse problem is solved with a standard weighted least-squares preconditioned conjugate gradient method (Tarantola, 1987) for elastic parameters reconstruction. The Born approxi-
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The model parameter \( \hat{H} \) and \( G \) in the model (equations 5 & 6). In order to compute in parallel forward problem solutions, namely the incident wavefields and operators are basically computed by a weighted product of the Two central ingredients of the FWI algorithm is the computation of the gradient of the objective function while the full solution of the forward problem is recomputed at each iteration of the inverse problem using the FV method in a way similar to the FD method proposed by Gelis et al. (2007). The weighted least-squares objective function is given by

\[
\mathcal{J}(m) = \Delta d^T \mathbf{W}_d \Delta d
\]

(3)

where \( \Delta d \) is the residuals for horizontal and vertical particle velocities (the difference between the observed data and the data computed with model \( m \)), the superscript \( \dagger \) indicates the adjoint (transpose conjugate) and \( \mathbf{W}_d \) is a weighting operator applied to the data. Minimization of the cost function leads to the following solution for the model perturbation \( \Delta m \) after preconditioning the gradient (Pratt et al., 1998; Operto et al., 2006).

\[
\Delta m_i = -\alpha (\hat{H}_i + \epsilon I)^{-1} S \partial \mathcal{J}(m)/\partial m_i
\]

(4)

where \( \partial \mathcal{J}(m)/\partial m_i \) is the gradient of the objective function defined by

\[
\partial \mathcal{J}(m)/\partial m_i = \mathbf{V}^T \left[ \frac{\partial \mathbf{A}^T}{\partial m_i} \right] \mathbf{A}^{-1} \mathbf{S}^T \partial \mathcal{J}(m)/\partial m_i
\]

(5)

and \( \hat{H}_i = \text{diag} \{ \mathbf{J}^T \mathbf{W}_d \mathbf{J} \} \) denotes the diagonal elements of the weighted approximate Hessian \( \hat{H}_i \) and \( \mathbf{J} \) denotes the sensitivity matrix. The detailed expression of the gradient of the objective function as a function of the elastic Green functions is provided in Gelis et al. (2007). One element of the sensitivity matrix is given by

\[
\mathbf{J}_{k(m,n),i} = \mathbf{V}_m^T \left[ \frac{\partial \mathbf{A}^T}{\partial m_i} \right] \mathbf{A}^{-1} \delta_n
\]

(6)

where \( k(m,n) \) denotes a source/receiver-component couple of the acquisition system, \( m \) and \( n \) denotes one shot and one receiver component respectively. \( \delta_n \) is a source vector describing one impulsive source in the direction of the receiver component \( n \).

The diagonal of the approximate Hessian \( \hat{H}_i \) provides a preconditioner of the gradient which properly scales the perturbation model (Shin et al., 2001). The damping parameter \( \epsilon \) is used to avoid numerical instabilities (i.e. division by zero). The matrix \( \mathbf{S} \) is a smoothing regularization operator implemented in the form of a 2D Gaussian spatial filter (Ravaut et al., 2004). The term \( \partial \mathbf{A}/\partial m_i \) is the radiation pattern of the diffraction by the model parameter \( m_i \) (Tarantola, 1986). The parameter \( \alpha \) is the descent steplength computed by a parabola fitting along the descent direction. The linearized inversion procedure is applied in cascade to several frequencies (or groups of discrete frequencies). For each frequency group, several iterations are computed. The use of conjugate directions of the gradient with Polak-Ribiere method (Mora, 1987) allows to speed up convergence of our algorithm.

**Parallel Implementation**

Two central ingredients of the FWI algorithm is the computation of the gradient \( \partial \mathcal{J}(m)/\partial m_i \) and its preconditioner \( \hat{H}_i \). These two operators are basically computed by a weighted product of the forward problem solutions, namely the incident wavefields and the backpropagated residual wavefields computed in the starting model (equations 5 & 6). In order to compute in parallel \( \partial \mathcal{J}(m)/\partial m_i \) and \( \hat{H}_i \), we take advantage of the distributed storage of the LU decomposition provided by the MUMPS solver: each processor stores a subdomain of LU factors of impedance matrix \( \mathbf{A} \) and all the corresponding partial solutions. Therefore, each processor computes a part of the gradient and of the preconditioner according to the domain decomposition of the distributed forward problem. Due to the non-diagonal pattern of derivative matrix \( \partial \mathbf{A}/\partial m_i \), point-to-point communications are required between processors to exchange solutions for all cells located at interfaces between subdomains. Figure 1 summarized the parallel FWI algorithm.

**VALIDATION TESTS**

In this section, we validate our algorithm with three simple numerical examples. The two first examples use only body waves (i.e. absorbing conditions are applied on the four edges of model) to image an elastic layer embedded in a homogeneous background from a multichannel seismic reflection acquisition and small P- and S-wave velocity (hereinafter referred to as \( V_P \) and \( V_S \)) anomalies from a crosshole geometry respectively. Aim of the third example is to reconstruct a \( V_P \) and \( V_S \) anomalies in a velocity gradient model below a Gaussian topography. All these examples were performed in regular equilateral meshes to guarantee accurate propagation modeling.

**Layer Model**

The Layer model of dimension \( 8 \times 3.5 \text{ km}^2 \) is composed of an homogeneous background with a 100-m thick horizontal layer. P-wave velocities are 3500 m/s and 3700 m/s in the background and in the layer respectively while the S-wave velocities are 2020 m/s and 2140 m/s respectively. Eighty vertical-force sources with a 80-m spacing are located 1300 m above the layer. The receiver array located 1400 m above the layer corresponds to 86 vertical and horizontal geophones with a 80-m spacing. Inversion for imaging \( V_P \) and \( V_S \) was sequentially applied to frequencies 1.96, 3.91, 5.87, 7.82, 9.8 and 12.7 Hz using the homogeneous background models as starting models. The frequency interval was chosen in order to preserve some wavenumber redundancy according to the Sirgue and Pratt (2004) rule. The gradient method without conjugate directions was used for this test. Ten iterations were computed per frequency. The correct \( V_P \) and \( V_S \) values were reconstructed in the layer with a resolution consistent with the frequency bandwidth involved in the inversion and the aperture illumination provided by the acquisition geometry (Figure 2). Oscillations parallel to the layer in the reconstructed models result from the narrow frequency bandwidth involved in the inversion.

**Crosshole test**

The crosshole test involves a 4000-m square homogeneous model with 3 anomalies (Figure 3). The model perturbation is composed of 250-m square positive \( V_P \) and \( V_S \) anomalies, of a 300-m circular negative \( V_P \) anomaly and of a 200-m circular negative \( V_S \) anomaly respectively. In each case, the amplitude of the \( V_P \) and \( V_S \) perturbations equals to \( \pm 8 \% \) of the \( V_P \) and \( V_S \) background velocities respectively. We used isolated \( V_P \) and \( V_S \) anomalies to assess the coupling between the two parameters during inversion. The crosshole acquisition corresponds to a vertical source line of 71 horizontal forces recorded by a
vertical line of 71 horizontal and vertical sensors (Figure 3). Inversion based on the gradient method was sequentially applied to frequencies 2.9, 4.9, 6.8, 9.8, 12.7 and 16.6 Hz for $V_p$ and $V_S$ parameters. The starting models are the homogeneous $V_p$ and $V_S$ background models. Ten iterations per frequency were computed. The reconstructed $V_p$ and $V_S$ models are shown in Figure 4. Figure 5 shows vertical profiles crossing anomalies at horizontal distances of 1410 m and 2500 m respectively. Heterogeneities are well located, focused and uncoupled: no perturbations are observed in the $V_p$ model at the location of the isolated $V_S$ anomaly and reciprocally. Resolution of the $V_S$ model is higher that of the $V_p$ one due to shorter propagated wavelengths. Limited-bandwidth effects, resulting from to the limited bandwidth of the source and the limited aperture illumination respectively, clearly affect the shape of the reconstructed anomalies and mainly correspond to a deficit of small wavenumbers for the vertical cross-section of the anomalies (Figure 5).

**Hill Model test**

The hill model is a 1D velocity gradient model below a topography with a Gaussian shape (Figure 6). The model perturbation is composed of a 300-m circular $V_p$ anomaly of amplitude -200 m/s and of a 200-m circular $V_S$ anomaly of amplitude -500 m/s representing two Poisson ratio ($\nu$) anomalies of 0.17 and 0.42 respectively with respect to the background model where $\nu = 0.25$. Eighty-six explosive sources were located 60 m below topography with a 80 m spacing and were recorded by sixty vertical and horizontal sensors located 30 m below topography. Inversion based on the conjugate gradient algorithm was sequentially applied to frequencies 2.9, 4.9, 6.8, 9.8, 12.7 and 16.6 Hz for $V_p$ and $V_S$ parameters. The starting model is the velocity gradient background. Ten iterations per frequency were computed. Reconstructed models are shown in Figure 7 where both $V_p$ and $V_S$ anomalies can be clearly observed with again a higher resolution for the $V_S$ anomaly. Vertical profiles across the $V_p$ and $V_S$ anomalies allow to assess the quantitative reconstruction of the elastic parameters (Figure 8). One can note a coupling between $V_p$ and $V_S$ parameters at the location of the $V_S$ anomaly where the reconstructed $V_p$ perturbations are not nil. Like in the previous example, we also observed a deficit of small wavenumbers in the shape of the anomalies along the vertical direction due to the surface-to-surface illumination.

**CONCLUSION**

We have implemented a massively parallel 2D frequency-domain full-waveform inversion algorithm for elastic parameters reconstruction based on a finite volume $P_1$ forward problem. We validated this algorithm against simple synthetic examples. Future works will both concern improvement of the forward problem algorithm and application of the inverse problem to realistic synthetic case studies. Concerning the forward problem, we will investigate frequency-domain $P_1$ Discontinuous-Galerkin method which theoretically gives the required accurate solutions of elastic wave propagation in unstructured meshes. We may adapt the mesh to the local propagated wavelength in order to reduce the size of linear systems to be solved. Concerning the inverse problem, applications to realistic synthetic models should be first tackled to better assess the sensitivity of the inversion to the acquisition geometry, to the choice of the elastic parameters and to the surface waves for onshore applications before considering application to real data.

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Figure 3: Crosshole test. True $V_P$ (a) and $V_S$ (b) models. S and R denote the source and receiver lines respectively.

Figure 4: Crosshole test. Reconstructed $V_P$ (a) and $V_S$ (b) models. Note the good uncoupling between $V_P$ and $V_S$ parameters.

Figure 5: Vertical profiles across anomalies at distances a) 1410 m and b) 2500 m. The true and reconstructed profiles are plotted with solid and dash lines respectively.

Figure 6: True Hill model. a) $V_P$ model b) $V_S$ model.

Figure 7: Reconstructed Hill models for a) $V_P$ and b) $V_S$ parameters.

Figure 8: $V_P$ and $V_S$ vertical profiles across the $V_P$ (left) and $V_S$ (right) anomalies. The true and reconstructed profiles are plotted with solid and dash lines respectively.
REFERENCES


